On integrability of the equations for nonsingular pairs of compatible flat metrics[1]

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1 Introduction. Basic definitions

In this paper, we deal with the problem of description of nonsingular pairs of compatible flat metrics for the general N-component case. We describe the scheme of the integrating the nonlinear equations describing nonsingular pairs of compatible flat metrics (or, in other words, nonsingular flat pencils of metrics). This scheme was announced in our previous paper [1]. It is based on the reducing this problem to a special reduction of the Lamé equations and the using the Zakharov method of differential reductions [2] in the dressing method (a version of the inverse scattering method).

We shall use both contravariant metrics $g^{ij}(u)$ with upper indices, where $u = (u^1, ..., u^N)$ are local coordinates, $1 \leq i, j \leq N$, and covariant metrics $g_{ij}(u)$ with lower indices, $g^{is}(u)g_{sj}(u) = \delta^i_j$. The indices of coefficients of the Levi–Civita connections $\Gamma^i_{jk}(u)$ (the Riemannian connections generated by the corresponding metrics) and tensors of Riemannian curvature $R^i_{jkl}(u)$ are raised and lowered by the metrics corresponding to them:

$$\Gamma^{ij}_k(u) = g^{is}(u)\Gamma^j_{sk}(u), \quad \Gamma^i_{jk}(u) = \frac{1}{2}g^{is}(u)\left(\frac{\partial g_{sk}}{\partial u^j} + \frac{\partial g_{js}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^s}\right),$$

$$R^{ij}_{kl}(u) = g^{is}(u)R^j_{skl}(u), \quad R^i_{jkl}(u) = -\frac{\partial \Gamma^i_{jl}}{\partial u^k} + \frac{\partial \Gamma^i_{jk}}{\partial u^l} - \Gamma^i_{pk}(u)\Gamma^p_{jl}(u) + \Gamma^i_{pl}(u)\Gamma^p_{jk}(u).$$

Definition 1.1 ([3], [4]) *Two contravariant flat metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ are called compatible if any linear combination of these metrics*

$$g^{ij}(u) = \lambda_1 g^{ij}_1(u) + \lambda_2 g^{ij}_2(u), \tag{1.1}$$

where λ_1 and λ_2 are arbitrary constants such that $\det(g^{ij}(u)) \not\equiv 0$, is also a flat metric and the coefficients of the corresponding Levi–Civita connections are related by the same linear formula:

$$\Gamma^{ij}_k(u) = \lambda_1 \Gamma^{ij}_{1,k}(u) + \lambda_2 \Gamma^{ij}_{2,k}(u). \tag{1.2}$$

We shall also say in this case that the metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ form a flat pencil.

Definition 1.2 ([1]) *Two contravariant metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ of constant Riemannian curvature K_1 and K_2, respectively, are called compatible if any linear combination of these metrics*

$$g^{ij}(u) = \lambda_1 g^{ij}_1(u) + \lambda_2 g^{ij}_2(u), \tag{1.3}$$

where λ_1 and λ_2 are arbitrary constants such that $\det(g^{ij}(u)) \not\equiv 0$, is a metric of constant Riemannian curvature $\lambda_1 K_1 + \lambda_2 K_2$ and the coefficients of the corresponding Levi–Civita connections are related by the same linear formula:

$$\Gamma_k^{ij}(u) = \lambda_1 \Gamma_{1,k}^{ij}(u) + \lambda_2 \Gamma_{2,k}^{ij}(u). \tag{1.4}$$

We shall also say in this case that the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ form a pencil of metrics of constant Riemannian curvature.

Definition 1.3 ([1]) *Two Riemannian or pseudo-Riemannian contravariant metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are called compatible if for any linear combination of these metrics*

$$g^{ij}(u) = \lambda_1 g_1^{ij}(u) + \lambda_2 g_2^{ij}(u), \tag{1.5}$$

where λ_1 and λ_2 are arbitrary constants such that $\det(g^{ij}(u)) \not\equiv 0$, the coefficients of the corresponding Levi–Civita connections and the components of the corresponding tensors of Riemannian curvature are related by the same linear formula:

$$\Gamma_k^{ij}(u) = \lambda_1 \Gamma_{1,k}^{ij}(u) + \lambda_2 \Gamma_{2,k}^{ij}(u), \tag{1.6}$$

$$R_{kl}^{ij}(u) = \lambda_1 R_{1,kl}^{ij}(u) + \lambda_2 R_{2,kl}^{ij}(u). \tag{1.7}$$

We shall also say in this case that the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ form a pencil of metrics.

Definition 1.4 ([1]) *Two Riemannian or pseudo-Riemannian contravariant metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are called almost compatible if for any linear combination of these metrics (1.5) relation (1.6) is fulfilled.*

Definition 1.5 *Two Riemannian or pseudo-Riemannian metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are called nonsingular pair of metrics if the eigenvalues of this pair of metrics, that is, the roots of the equation*

$$\det(g_1^{ij}(u) - \lambda g_2^{ij}(u)) = 0, \tag{1.8}$$

are distinct.

These definitions are motivated by the theory of compatible Poisson brackets of hydrodynamic type. In the case if the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are flat, that is, $R_{1,jkl}^i(u) = R_{2,jkl}^i(u) = 0$, relation (1.7) is equivalent to the condition that an arbitrary linear combination of the flat metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ is also a flat metric and Definition 1.3 is equivalent to the well-known definition of a flat pencil of metrics or, in other words, a compatible pair of local nondegenerate Poisson structures of hydrodynamic type [3] (see also [4]–[10]). In the case if the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are metrics of constant Riemannian curvature K_1 and K_2, respectively, that is,

$$R_{1,kl}^{ij}(u) = K_1(\delta_k^i \delta_l^j - \delta_l^i \delta_k^j), \quad R_{2,kl}^{ij}(u) = K_2(\delta_k^i \delta_l^j - \delta_l^i \delta_k^j),$$

relation (1.7) gives the condition that an arbitrary linear combination of the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ (1.5) is a metric of constant Riemannian curvature $\lambda_1 K_1 + \lambda_2 K_2$ and Definition 1.3 is equivalent to Definition 1.1 of a pencil of metrics of constant Riemannian curvature or, in other

words, a compatible pair of the corresponding nonlocal Poisson structures of hydrodynamic type which were introduced and studied by the author and Ferapontov in [12]. Compatible metrics of more general type correspond to compatible pairs of nonlocal Poisson structures of hydrodynamic type which were introduced and studied by Ferapontov in [13]. They arise, for example, if we shall use a recursion operator generated by a pair of compatible Poisson structures of hydrodynamic type and determining, as is well-known, an infinite sequence of corresponding Poisson structures.

2 Compatible local Poisson structures of hydrodynamic type (a brief survey)

The local homogeneous Poisson bracket of the first order, that is, the Poisson bracket of the form

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\,\delta_x(x-y) + b_k^{ij}(u(x))\,u_x^k\,\delta(x-y), \qquad (2.1)$$

where $u^1, ..., u^N$ are local coordinates on a certain smooth N-dimensional manifold M, is called a *local Poisson structure of hydrodynamic type* or *Dubrovin–Novikov structure* [11]. Here, $u^i(x)$, $1 \le i \le N$, are functions (fields) of a single independent variable x, and the coefficients $g^{ij}(u)$ and $b_k^{ij}(u)$ of bracket (2.1) are smooth functions on M.

In other words, for arbitrary functionals $I[u]$ and $J[u]$ on the space of fields $u^i(x)$, $1 \le i \le N$, a bracket of the form

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \left(g^{ij}(u(x)) \frac{d}{dx} + b_k^{ij}(u(x))\,u_x^k \right) \frac{\delta J}{\delta u^j(x)}\,dx \qquad (2.2)$$

is defined and it is required that this bracket is a Poisson bracket, that is, it is skew-symmetric:

$$\{I, J\} = -\{J, I\}, \qquad (2.3)$$

and satisfies the Jocobi identity

$$\{\{I, J\}, K\} + \{\{J, K\}, I\} + \{\{K, I\}, J\} = 0 \qquad (2.4)$$

for arbitrary functionals $I[u]$, $J[u]$ and $K[u]$. The skew-symmetry (2.3) and the Jacobi identity (2.4) impose very strict conditions on the coefficients $g^{ij}(u)$ and $b_k^{ij}(u)$ of bracket (2.2) (these conditions will be considered below).

The local Poisson structures of hydrodynamic type (2.1) were introduced and studied by Dubrovin and Novikov in [11]. In this paper, they proposed a general Hamiltonian approach to the so-called *homogeneous systems of hydrodynamic type*, that is, to evolutionary quasilinear systems of first-order partial differential equations

$$u_t^i = V_j^i(u)\,u_x^j \qquad (2.5)$$

that corresponds to structures (2.1).

This Hamiltonian approach was motivated by the study of the equations of Euler hydrodynamics and the Whitham averaging equations that describe the evolution of slowly modulated multiphase solutions of partial differential equations [17].

Local bracket (2.2) is called *nondegenerate* if $\det(g^{ij}(u)) \not\equiv 0$. For general nondegenerate brackets of form (2.2), Dubrovin and Novikov proved the following important theorem.

Theorem 2.1 (Dubrovin, Novikov [11]) *If $\det(g^{ij}(u)) \not\equiv 0$, then bracket (2.2) is a Poisson bracket, that is, it is skew-symmetric and satisfies the Jacobi identity if and only if*

(1) *$g^{ij}(u)$ is an arbitrary flat pseudo-Riemannian contravariant metric (a metric of zero Riemannian curvature),*

(2) *$b_k^{ij}(u) = -g^{is}(u)\Gamma_{sk}^j(u)$, where $\Gamma_{sk}^j(u)$ is the Riemannian connection generated by the contravariant metric $g^{ij}(u)$ (the Levi–Civita connection).*

Consequently, for any local nondegenerate Poisson structure of hydrodynamic type, there always exist local coordinates $v^1, ..., v^N$ (flat coordinates of the metric $g^{ij}(u)$) in which the coefficients of the brackets are constant:

$$\widetilde{g}^{ij}(v) = \eta^{ij} = \text{ const}, \ \ \widetilde{\Gamma}_{jk}^i(v) = 0, \ \ \widetilde{b}_k^{ij}(v) = 0, \tag{2.6}$$

that is, the bracket has the constant form

$$\{I, J\} = \int \frac{\delta I}{\delta v^i(x)} \eta^{ij} \frac{d}{dx} \frac{\delta J}{\delta v^j(x)} dx, \tag{2.7}$$

where (η^{ij}) is a nondegenerate symmetric constant matrix:

$$\eta^{ij} = \eta^{ji}, \ \ \eta^{ij} = \text{const}, \ \ \det(\eta^{ij}) \neq 0. \tag{2.8}$$

On the other hand, as early as 1978, Magri proposed a bi-Hamiltonian approach to the integration of nonlinear systems [18]. This approach demonstrated that the integrability is closely related to the bi-Hamiltonian property, that is, to the property of a system to have two compatible Hamiltonian representations. As was shown by Magri in [18], compatible Poisson brackets generate integrable hierarchies of systems of differential equations. Therefore, the description of compatible Poisson structures is very urgent and important problem in the theory of integrable systems. In particular, for a system, the bi-Hamiltonian property generates recurrent relations for the conservation laws of this system.

Beginning from [18], quite extensive literature (see, for example, [19]–[23] and the necessary references therein) has been devoted to the bi-Hamiltonian approach and to the construction of compatible Poisson structures for many specific important equations of mathematical physics and field theory. As far as the problem of description of sufficiently wide classes of compatible Poisson structures of defined special types is concerned, apparently the first such statement was considered in [24], [25] (see also [26], [27]). In those papers, the present author posed and completely solved the problem of description of all compatible local scalar first-order and third-order Poisson brackets, that is, all Poisson brackets given by arbitrary scalar first-order and third-order ordinary differential operators. These brackets generalize the well-known compatible pair of the Gardner–Zakharov–Faddeev bracket [28], [29] (first-order bracket) and the Magri bracket [18] (third-order bracket) for the Korteweg–de Vries equation.

In the case of homogeneous systems of hydrodynamic type, many integrable systems possess compatible Poisson structures of hydrodynamic type. The problems of description of these structures for particular systems and numerous examples were considered in many papers (see, for example, [30]–[35]). In particular, in [30] Nutku studied a special class of compatible two-component Poisson structures of hydrodynamic type and the related bi-Hamiltonian hydrodynamic systems. In [14] Ferapontov classified all two-component homogeneous systems of hydrodynamic type possessing three compatible local Poisson structures of hydrodynamic type.

In the general form, the problem of description of flat pencil of metrics (or, in other words, compatible nondegenerate local Poisson structures of hydrodynamic type) was considered by Dubrovin in [3], [4] in connection with the construction of important examples of such flat pencils of metrics, generated by natural pairs of flat metrics on the spaces of orbits of Coxeter groups and on other Frobenius manifolds and associated with the corresponding quasi-homogeneous solutions of the associativity equations. In the theory of Frobenius manifolds introduced and studied by Dubrovin [3], [4] (they correspond to two-dimensional topological field theories), a key role is played by flat pencils of metrics, possessing a number of special additional (and very strict) properties (they satisfy the so-called quasi-homogeneity property). In addition, in [5] Dubrovin proved that the theory of Frobenius manifolds is equivalent to the theory quasi-homogeneous compatible nondegenerate Poisson structures of hydrodynamic type. The general problem of compatible nondegenerate local Poisson structures was also considered by Ferapontov in [6].

The author's papers [7]–[10] are devoted to the general problem of classification of local Poisson structures of hydrodynamic type, to integrable nonlinear systems which describe such compatible Poisson structures and to special reductions connected with the associativity equations.

Definition 2.1 (Magri [18]) *Two Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ are called compatible if an arbitrary linear combination of these Poisson brackets*

$$\{\ ,\ \} = \lambda_1 \{\ ,\ \}_1 + \lambda_2 \{\ ,\ \}_2, \tag{2.9}$$

where λ_1 and λ_2 are arbitrary constants, is also always a Poisson bracket. In this case, one can say also that the brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ form a pencil of Poisson brackets.

Correspondingly, the problem of description of compatible nondegenerate local Poisson structures of hydrodynamic type is pure differential-geometric problem of description of flat pencils of metrics (see [3], [4]).

In [3], [4] Dubrovin presented all the tensor relations for the general flat pencils of metrics. First, we introduce the necessary notation. Let ∇_1 and ∇_2 be the operators of covariant differentiation given by the Levi–Civita connections $\Gamma_{1,k}^{ij}(u)$ and $\Gamma_{2,k}^{ij}(u)$, generated by the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$, respectively. The indices of the covariant differentials are raised and lowered by the corresponding metrics: $\nabla_1^i = g_1^{is}(u)\nabla_{1,s}$, $\nabla_2^i = g_2^{is}(u)\nabla_{2,s}$. Consider the tensor

$$\Delta^{ijk}(u) = g_1^{is}(u)g_2^{jp}(u)\left(\Gamma_{2,ps}^k(u) - \Gamma_{1,ps}^k(u)\right), \tag{2.10}$$

introduced by Dubrovin in [3], [4].

Theorem 2.2 (Dubrovin [3], [4]) *If metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ form a flat pencil, then there exists a vector field $f^i(u)$ such that the tensor $\Delta^{ijk}(u)$ and the metric $g_1^{ij}(u)$ have the form*

$$\Delta^{ijk}(u) = \nabla_2^i \nabla_2^j f^k(u), \tag{2.11}$$

$$g_1^{ij}(u) = \nabla_2^i f^j(u) + \nabla_2^j f^i(u) + cg_2^{ij}(u), \tag{2.12}$$

where c is a certain constant, and the vector field $f^i(u)$ satisfies the equations

$$\Delta_s^{ij}(u)\Delta_l^{sk}(u) = \Delta_s^{ik}(u)\Delta_l^{sj}(u), \tag{2.13}$$

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where

$$\Delta_k^{ij}(u) = g_{2,ks}(u)\Delta^{sij}(u) = \nabla_{2,k}\nabla_2^i f^j(u), \tag{2.14}$$

and

$$(g_1^{is}(u)g_2^{jp}(u) - g_2^{is}(u)g_1^{jp}(u))\nabla_{2,s}\nabla_{2,p}f^k(u) = 0. \tag{2.15}$$

Conversely, for the flat metric $g_2^{ij}(u)$ and the vector field $f^i(u)$ that is a solution of the system of equations (2.13) and (2.15), the metrics $g_2^{ij}(u)$ and (2.12) form a flat pencil.

The proof of this theorem immediately follows from the relations that are equivalent to the fact that the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ form a flat pencil and are considered in flat coordinates of the metric $g_2^{ij}(u)$ [3], [4].

In my paper [7], an explicit and simple *criterion* of compatibility for two Poisson structures of hydrodynamic type is formulated, that is, it is shown what explicit form is sufficient and necessary for the Poisson structures of hydrodynamic type to be compatible.

For the moment, we are able to formulate such explicit general criterion only namely in terms of Poisson structures but not in terms of metrics as in Theorem 2.2.

Lemma 2.1 ([7]) (An explicit criterion of compatibility for Poisson structures of hydrodynamic type) *Any local Poisson structure of hydrodynamic type $\{I, J\}_2$ is compatible with the constant nondegenerate Poisson bracket (2.7) if and only if it has the form*

$$\{I, J\}_2 = \int \frac{\delta I}{\delta v^i(x)} \left(\left(\eta^{is}\frac{\partial h^j}{\partial v^s} + \eta^{js}\frac{\partial h^i}{\partial v^s} \right) \frac{d}{dx} + \eta^{is}\frac{\partial^2 h^j}{\partial v^s \partial v^k}v_x^k \right) \frac{\delta J}{\delta v^j(x)}dx, \tag{2.16}$$

where $h^i(v)$, $1 \leq i \leq N$, are smooth functions defined on a certain neighbourhood.

We do not require in Lemma 2.1 that the Poisson structure of hydrodynamic type $\{I, J\}_2$ is nondegenerate. Besides, it is important to note that this statement is local.

In 1995, in the paper [6], Ferapontov proposed an approach to the problem on flat pencils of metrics, which is motivated by the theory of recursion operators, and formulated the following theorem as a criterion of compatibility of nondegenerate local Poisson structures of hydrodynamic type:

Theorem 2.3 ([6]) *Two local nondegenerate Poisson structures of hydrodynamic type given by flat metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are compatible if and only if the Nijenhuis tensor of the affinor $v_j^i(u) = g_1^{is}(u)g_{2,sj}(u)$ vanishes, that is,*

$$N_{ij}^k(u) = v_i^s(u)\frac{\partial v_j^k}{\partial u^s} - v_j^s(u)\frac{\partial v_i^k}{\partial u^s} + v_s^k(u)\frac{\partial v_i^s}{\partial u^j} - v_s^k\frac{\partial v_j^s}{\partial u^i} = 0. \tag{2.17}$$

Besides, it is noted in the remark in [6] that if the spectrum of $v_j^i(u)$ is simple, the vanishing of the Nijenhuis tensor implies the existence of coordinates $R^1, ..., R^N$ for which all the objects $v_j^i(u)$, $g_1^{ij}(u)$, $g_2^{ij}(u)$ become diagonal. Moreover, in these coordinates the ith eigenvalue of $v_j^i(u)$ depends only on the coordinate R^i. In the case when all the eigenvalues are nonconstant, they can be introduced as new coordinates. In these new coordinates $\tilde{v}_j^i(R) = \text{diag}\,(R^1, ..., R^N)$, $\tilde{g}_2^{ij}(R) = \text{diag}\,(g^1(R), ..., g^N(R))$, $\tilde{g}_1^{ij}(R) = \text{diag}\,(R^1 g^1(R), ..., R^N g^N(R))$.

Unfortunately, as is shown in [1], in the general case the Theorem 2.3 is not true and, correspondingly, it is not a criterion of compatibility of flat metrics. Generally speaking, compatibility of flat metrics does not follow from the vanishing of the corresponding Nijenhuis tensor. The

corresponding counterexamples were presented in [1]. In the general case, as it was shown in [1], the Theorem 2.3 is actually a criterion of almost compatibility of flat metrics that does not guarantee compatibility of the corresponding nondegenerate local Poisson structures of hydrodynamic type. But if the spectrum of $v_j^i(u)$ is simple, that is, all the eigenvalues are distinct, then the Theorem 2.3 is not only true but also can be essentially generalized for the case of arbitrary compatible Riemannian or pseudo-Riemannian metrics, in particular, for the especially important cases in the theory of systems of hydrodynamiic type, namely, the cases of metrics of constant Riemannian curvature or the metrics generating the general nonlocal Poisson structures of hydrodynamic type (see [1]).

In particular, the following general theorem is proved in [1]:

Theorem 2.4 ([1]) *An arbitrary nonsingular pair of metrics is compatible if and only if there exist local coordinates $u = (u^1, ..., u^N)$ such that $g_2^{ij}(u) = g^i(u)\delta^{ij}$ and $g_1^{ij}(u) = f^i(u^i)g^i(u)\delta^{ij}$, where $f^i(u^i)$, $i = 1, ..., N$, are arbitrary functions of single variables (of course, in the case of nonsingular pair of metrics, these functions are not equal to each other if they are constants and they are not equal identically to zero).*

In this paper, we consider only the case of nonsingular pairs of flat metrics. In this case the approach of Ferapontov and Theorem 2.3 are absolutely correct.

3 Equations for nonsingular pairs of compatible flat metrics

Let us consider here the problem on nonsingular pairs of compatible flat metrics. It follows from Theorem 2.3 and Theorem 2.4 that it is sufficient to classify flat metrics of the form $g_2^{ij}(u) = g^i(u)\delta^{ij}$ and $g_1^{ij}(u) = f^i(u^i)g^i(u)\delta^{ij}$, where $f^i(u^i)$, $i = 1, ..., N$, are arbitrary functions of single variables.

The problem of description of diagonal flat metrics, that is, flat metrics $g_2^{ij}(u) = g^i(u)\delta^{ij}$, is a classical problem of differential geometry. This problem is equivalent to the problem of description of curvilinear orthogonal coordinate systems in a pseudo-Euclidean space and it was studied in detail and mainly solved in the beginning of the 20th century (see [38]). Locally, such coordinate systems are determined by $n(n-1)/2$ arbitrary functions of two variables (see [39], [40]). Recently, Zakharov showed that the Lamé equations describing curvilinear orthogonal coordinate systems can be integrated by the inverse scattering method [2] (see also an algebraic-geometric approach in [41]).

Theorem 3.1 *Nonsingular pairs of compatible flat metrics are described by the following integrable nonlinear systems which are the special reductions of the Lamé equations:*

$$\frac{\partial \beta_{ij}}{\partial u^k} = \beta_{ik}\beta_{kj}, \quad i \neq j, \quad i \neq k, \quad j \neq k, \tag{3.1}$$

$$\frac{\partial \beta_{ij}}{\partial u^i} + \frac{\partial \beta_{ji}}{\partial u^j} + \sum_{s \neq i, \ s \neq j} \beta_{si}\beta_{sj} = 0, \quad i \neq j, \tag{3.2}$$

$$\sqrt{f^i(u^i)}\frac{\partial\left(\sqrt{f^i(u^i)}\beta_{ij}\right)}{\partial u^i} + \sqrt{f^j(u^j)}\frac{\partial\left(\sqrt{f^j(u^j)}\beta_{ji}\right)}{\partial u^j} + \sum_{s \neq i, \ s \neq j} f^s(u^s)\beta_{si}\beta_{sj} = 0, \quad i \neq j, \tag{3.3}$$

where $f^i(u^i)$, $i = 1, ..., N$, are the given arbitrary functions of single variables.

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The equations (3.1) and (3.2) are the famous Lamé equations and the equation (3.3) defines a nontrivial nonlinear differential reduction of the Lamé equations. Such types of differential reductions for Lamé equations were also studied by Zakharov [2] and the Zakharov method can be applied successfully to our problem.

Consider the conditions of flatness for the diagonal metrics $g_2^{ij}(u) = g^i(u)\delta^{ij}$ and $g_1^{ij}(u) = f^i(u^i)g^i(u)\delta^{ij}$, where $f^i(u^i)$, $i = 1, ..., N$, are arbitrary functions of the given single variables (but these functions are not equal to zero identically).

Recall that for any diagonal metric $\Gamma^i_{jk}(u) = 0$ if all the indices i, j, k are distinct. Correspondingly, $R^{ij}_{kl}(u) = 0$ if all the indices i, j, k, l are distinct. Besides, as a result of the well-known symmetries of the tensor of Riemannian curvature we have:

$$R^{ii}_{kl}(u) = R^{ij}_{kk}(u) = 0,$$

$$R^{ij}_{il}(u) = -R^{ij}_{li}(u) = R^{ji}_{li}(u) = -R^{ji}_{il}(u).$$

Thus, it is sufficient to consider the condition $R^{ij}_{kl}(u) = 0$ (the condition of the flatness for a metric) only for the following components of the tensor of Riemannian curvature: $R^{ij}_{il}(u)$, where $i \neq j$, $i \neq l$.

For any diagonal metric $g_2^{ij}(u) = g^i(u)\delta^{ij}$ we have

$$\Gamma^i_{2,ik}(u) = \Gamma^i_{2,ki}(u) = -\frac{1}{2g^i(u)}\frac{\partial g^i}{\partial u^k}, \quad \text{for any } i, k;$$

$$\Gamma^i_{2,jj}(u) = \frac{1}{2}\frac{g^i(u)}{(g^j(u))^2}\frac{\partial g^j}{\partial u^i}, \quad i \neq j.$$

$$R^{ij}_{2,il}(u) = g^i(u)R^j_{2,iil}(u) =$$
$$g^i(u)\left(-\frac{\partial \Gamma^j_{2,il}}{\partial u^i} + \frac{\partial \Gamma^j_{2,ii}}{\partial u^l} - \sum_{s=1}^{N}\Gamma^j_{2,si}(u)\Gamma^s_{2,il}(u) + \sum_{s=1}^{N}\Gamma^j_{2,sl}(u)\Gamma^s_{2,ii}(u)\right). \qquad (3.4)$$

It is necessary to consider separately two different cases.
1) $j \neq l$.

$$R^{ij}_{2,il}(u) = g^i(u)\left(\frac{\partial \Gamma^j_{2,ii}}{\partial u^l} - \Gamma^j_{2,ii}(u)\Gamma^i_{2,il}(u) + \Gamma^j_{2,jl}(u)\Gamma^j_{2,ii}(u) + \Gamma^l_{2,ll}(u)\Gamma^l_{2,ii}(u)\right) =$$

$$\frac{1}{2}g^i(u)\frac{\partial}{\partial u^l}\left(\frac{g^j(u)}{(g^i(u))^2}\frac{\partial g^i}{\partial u^j}\right) + \frac{1}{4}\frac{g^j(u)}{(g^i(u))^2}\frac{\partial g^i}{\partial u^j}\frac{\partial g^i}{\partial u^l}$$

$$-\frac{1}{4g^i(u)}\frac{\partial g^i}{\partial u^j}\frac{\partial g^j}{\partial u^l} + \frac{1}{4}\frac{g^j(u)}{g^i(u)g^l(u)}\frac{\partial g^l}{\partial u^j}\frac{\partial g^i}{\partial u^l} = 0. \qquad (3.5)$$

Introducing the standard classic notation

$$g^i(u) = \frac{1}{(H_i(u))^2}, \quad ds^2 = \sum_{i=1}^{N}(H_i(u))^2(du^i)^2, \qquad (3.6)$$

$$\beta_{ik}(u) = \frac{1}{H_i(u)}\frac{\partial H_k}{\partial u^i}, \quad i \neq k, \qquad (3.7)$$

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where $H_i(u)$ are the *Lamé coefficients* and $\beta_{ik}(u)$ are the rotation coefficients, we obtain that equations (3.5) are equivalent to the equations

$$\frac{\partial^2 H_i}{\partial u^j \partial u^k} = \frac{1}{H_j(u)} \frac{\partial H_i}{\partial u^j} \frac{\partial H_j}{\partial u^k} + \frac{1}{H_k(u)} \frac{\partial H_k}{\partial u^j} \frac{\partial H_i}{\partial u^k} \tag{3.8}$$

or, equivalently, to equations (3.1).

2) $j = l$.

$$R_{2,ij}^{ij}(u) = g^i(u) \left(-\frac{\partial \Gamma_{2,ij}^j}{\partial u^i} + \frac{\partial \Gamma_{2,ii}^j}{\partial u^j} - \Gamma_{2,ii}^j(u) \Gamma_{2,ij}^i(u) - \right.$$

$$\left. \Gamma_{2,ji}^j(u) \Gamma_{2,ij}^j(u) + \sum_{s=1}^{N} \Gamma_{2,sj}^j(u) \Gamma_{2,ii}^s(u) \right) =$$

$$\frac{1}{2} g^i(u) \frac{\partial}{\partial u^i} \left(\frac{1}{g^j(u)} \frac{\partial g^j}{\partial u^i} \right) + \frac{1}{2} g^i(u) \frac{\partial}{\partial u^j} \left(\frac{g^j(u)}{(g^i(u))^2} \frac{\partial g^i}{\partial u^j} \right) + \frac{1}{4} \frac{g^j(u)}{(g^i(u))^2} \frac{\partial g^i}{\partial u^j} \frac{\partial g^i}{\partial u^j}$$

$$-\frac{1}{4} \frac{g^i(u)}{(g^j(u))^2} \frac{\partial g^j}{\partial u^i} \frac{\partial g^j}{\partial u^i} + \frac{1}{4g^j(u)} \frac{\partial g^j}{\partial u^i} \frac{\partial g^i}{\partial u^i} - \sum_{s \neq i} \frac{1}{4} \frac{g^s(u)}{g^i(u) g^j(u)} \frac{\partial g^j}{\partial u^s} \frac{\partial g^i}{\partial u^s} = 0. \tag{3.9}$$

Equations (3.9) are equivalent to the equations

$$\frac{\partial}{\partial u^i} \left(\frac{1}{H_i(u)} \frac{\partial H_j}{\partial u^i} \right) + \frac{\partial}{\partial u^j} \left(\frac{1}{H_j(u)} \frac{\partial H_i}{\partial u^j} \right) + \sum_{s \neq i,\ s \neq j} \frac{1}{(H_s(u))^2} \frac{\partial H_i}{\partial u^s} \frac{\partial H_j}{\partial u^s}, \quad i \neq j, \tag{3.10}$$

or, equivalently, to equations (3.2).

The condition that the metric $g_1^{ij}(u) = f^i(u^i) g^i(u) \delta^{ij}$ is also flat gives exactly $n(n-1)/2$ additional equations (3.3) which are linear with respect to the given functions $f^i(u^i)$. Note that, in this case, components (3.5) of tensor of Riemannian curvature automatically vanish. And the vanishing of components (3.9) gives the corresponding $n(n-1)/2$ equations.

Actually, for the metric $g_1^{ij}(u) = f^i(u^i) g^i(u) \delta^{ij}$, we have

$$\widetilde{H}_i(u) = \frac{H_i(u)}{\sqrt{f^i(u^i)}}, \quad \widetilde{\beta}_{ik}(u) = \frac{\sqrt{f^i(u^i)}}{\sqrt{f^k(u^k)}} \left(\frac{1}{H_i(u)} \frac{\partial H_k}{\partial u^i} \right) = \frac{\sqrt{f^i(u^i)}}{\sqrt{f^k(u^k)}} \beta_{ik}(u). \tag{3.11}$$

Respectively, equations (3.1) are also fulfilled for the rotation coefficients $\widetilde{\beta}_{ik}(u)$ and equations (3.2) for them give equation (3.3).

In particular, in the case $N = 2$ this completely solves the problem of description of nonsingular pairs of compatible flat metrics [1]. In the next section we give their complete description.

4   Two-component compatible flat metrics

We present here the complete description of nonsingular pairs of two-component compatible flat metrics [1] (see also [7], [9], [10], where an integrable four-component homogeneous system of hydrodynamic type, describing all the two-component compatible flat metrics, was derived and investigated).

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It is shown above that for any nonsingular pair of two-component compatible metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ there always exist local coordinates $u^1, ..., u^N$ such that

$$(g_2^{ij}(u)) = \begin{pmatrix} \frac{\varepsilon^1}{(b^1(u))^2} & 0 \\ 0 & \frac{\varepsilon^2}{(b^2(u))^2} \end{pmatrix}, \qquad (g_1^{ij}(u)) = \begin{pmatrix} \frac{\varepsilon^1 f^1(u^1)}{(b^1(u))^2} & 0 \\ 0 & \frac{\varepsilon^2 f^2(u^2)}{(b^2(u))^2} \end{pmatrix}, \qquad (4.1)$$

where $\varepsilon^i = \pm 1$, $i = 1, 2$; $b^i(u)$ and $f^i(u^i)$, $i = 1, 2$, are arbitrary nonzero functions of the corresponding single variables.

Lemma 4.1 *An arbitrary diagonal metric $g_2^{ij}(u)$ (4.1) is flat if and only if the functions $b^i(u)$, $i = 1, 2$, are solutions of the following linear system:*

$$\frac{\partial b^2}{\partial u^1} = \varepsilon^1 \frac{\partial F}{\partial u^2} b^1(u), \qquad \frac{\partial b^1}{\partial u^2} = -\varepsilon^2 \frac{\partial F}{\partial u^1} b^2(u), \qquad (4.2)$$

where $F(u)$ is an arbitrary function.

Theorem 4.1 ([1]) *The metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ (4.1) form a flat pencil of metrics if and only if the functions $b^i(u)$, $i = 1, 2$, are solutions of the linear system (4.2), where the function $F(u)$ is a solution of the following linear equation:*

$$2 \frac{\partial^2 F}{\partial u^1 \partial u^2}(f^1(u^1) - f^2(u^2)) + \frac{\partial F}{\partial u^2} \frac{df^1(u^1)}{du^1} - \frac{\partial F}{\partial u^1} \frac{df^2(u^2)}{du^2} = 0. \qquad (4.3)$$

In the case, if the eigenvalues of the pair of the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ are not only distinct but also are not constants, we can always choose local coordinates such that $f^1(u^1) = u^1$, $f^2(u^2) = u^2$ (see also remark in [6]). In this case, equation (4.3) has the form

$$2 \frac{\partial^2 F}{\partial u^1 \partial u^2}(u^1 - u^2) + \frac{\partial F}{\partial u^2} - \frac{\partial F}{\partial u^1} = 0. \qquad (4.4)$$

Let us continue this recurrent procedure for the metrics $G_{n+1}^{ij}(u) = v_j^i(u) G_n^{ij}(u)$ with the help of the affinor $v_j^i(u) = u^i \delta_j^i$.

Theorem 4.2 ([1]) *Three metrics*

$$(G_n^{ij}(u)) = \begin{pmatrix} \frac{\varepsilon^1 (u^1)^n}{(b^1(u))^2} & 0 \\ 0 & \frac{\varepsilon^2 (u^2)^n}{(b^2(u))^2} \end{pmatrix}, \qquad n = 0, 1, 2, \qquad (4.5)$$

form a flat pencil of metrics (pairwise compatible) if and only if the functions $b^i(u)$, $i = 1, 2$, are solutions of the linear system (4.2), where

$$F(u) = c \ln(u^1 - u^2), \qquad (4.6)$$

c is an arbitrary constant. Already the metric $G_3^{ij}(u)$ is flat only in the most trivial case, when $c = 0$, and, respectively, $b^1 = b^1(u^1)$, $b^2 = b^2(u^2)$.

The metric $G_3^{ij}(u)$ is a metric of nonzero constant Riemannian curvature $K \neq 0$ (in this case, the metrics G_n^{ij}, $n = 0, 1, 2, 3$, form a pencil of metrics of constant Riemannian curvature) if and only if

$$(b^1(u))^2 = (b^2(u))^2 = \frac{\varepsilon^2}{4K}(u^1 - u^2), \qquad \varepsilon^1 = -\varepsilon^2, \qquad c = \pm \frac{1}{2}. \qquad (4.7)$$

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5 Compatible flat metrics and the Zakharov method of differential reductions

Recall very briefly the Zakharov method of the integrating the Lamé equations (3.1) and (3.2) [2].

We should choose a matrix function $F_{ij}(s, s', u)$ and solve the integral equation

$$K_{ij}(s, s', u) = F_{ij}(s, s', u) + \int_s^\infty \sum_l K_{il}(s, q, u) F_{lj}(q, s', u) dq. \tag{5.1}$$

Then we obtain a one-parameter family of solutions of the Lamé equations by the formula

$$\beta_{ij}(s, u) = K_{ji}(s, s, u). \tag{5.2}$$

In particular, if $F_{ij}(s, s', u) = f_{ij}(s - u^i, s' - u^j)$, where $f_{ij}(x, y)$ is an arbitrary matrix function of two variables, then formula (5.2) produces solutions of equations (3.1). To satisfy equations (3.2), Zakharov proposed to impose on the "dressing matrix function" $F_{ij}(s - u^i, s' - u^j)$ a certain additional differential relation. If $F_{ij}(s - u^i, s' - u^j)$ satisfy the Zakharov differential relation, then the rotation coefficients $\beta_{ij}(u)$ satisfy additionally equations (3.2).

Lemma 5.1 *If both the function $F_{ij}(s - u^i, s' - u^j)$ and the function*

$$\widetilde{F}_{ij}(s - u^i, s' - u^j) = \frac{\sqrt{f^j(u^j - s')}}{\sqrt{f^i(u^i - s)}} F_{ij}(s - u^i, s' - u^j) \tag{5.3}$$

satisfy the Zakharov differential relation, then the corresponding rotation coefficients $\beta_{ij}(u)$ (5.2) satisfy both equations (3.2) and (3.3).

Actually, if $K_{ij}(s, s', u)$ is the solution of the integral equation (5.1) corresponding to the function $F_{ij}(s - u^i, s' - u^j)$, then

$$\widetilde{K}_{ij}(s, s', u) = \frac{\sqrt{f^j(u^j - s')}}{\sqrt{f^i(u^i - s)}} K_{ij}(s, s', u) \tag{5.4}$$

is the solution of (5.1) corresponding to function (5.3). It is simple to prove multiplying the integral equation (5.1) by $\sqrt{f^j(u^j - s')}/\sqrt{f^i(u^i - s)}$:

$$\widetilde{K}_{ij}(s, s', u) = \widetilde{F}_{ij}(s - u^i, s' - u^j) + \int_s^\infty \sum_l \widetilde{K}_{il}(s, q, u) \widetilde{F}_{lj}(q - u^l, s' - u^j) dq. \tag{5.5}$$

Then both $\widetilde{\beta}_{ij}(s, u) = \widetilde{K}_{ji}(s, s, u)$ and $\beta_{ij}(s, u) = K_{ji}(s, s, u)$ satisfy the Lamé equations (3.1) and (3.2). Besides, we have

$$\widetilde{\beta}_{ij}(s, u) = \widetilde{K}_{ji}(s, s, u) = \frac{\sqrt{f^i(u^i - s)}}{\sqrt{f^j(u^j - s)}} K_{ji}(s, s, u) = \frac{\sqrt{f^i(u^i - s)}}{\sqrt{f^j(u^j - s)}} \beta_{ij}(s, u). \tag{5.6}$$

Thus, in this case the rotation coefficients $\beta_{ij}(u)$ satisfy exactly all the equations (3.1)–(3.3), that is, they generate the corresponding compatible flat metrics.

The Zakharov differential reduction can be written as follows [2]:

$$\frac{\partial F_{ij}(s, s', u)}{\partial s'} + \frac{\partial F_{ji}(s', s, u)}{\partial s} = 0. \tag{5.7}$$

Thus, to resolve them for the matrix function $F_{ij}(s - u^i, s' - u^j)$, we can introduce $n(n-1)/2$ arbitrary functions of two variables $\Phi(x, y)$, $i < j$, and put for $i < j$

$$F_{ij}(s - u^i, s' - u^j) = \frac{\partial \Phi_{ij}(s - u^i, s' - u^j)}{\partial s},$$

$$F_{ji}(s - u^i, s' - u^j) = -\frac{\partial \Phi_{ij}(s' - u^i, s - u^j)}{\partial s}, \tag{5.8}$$

and

$$F_{ii}(s - u^i, s' - u^j) = \frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s}, \tag{5.9}$$

where $\Phi_{ii}(x, y)$, $i = 1, ..., N$, are arbitrary skew-symmetric functions:

$$\Phi_{ii}(x, y) = -\Phi_{ii}(y, x), \tag{5.10}$$

see [2].

For the function

$$\widetilde{F}_{ij}(s - u^i, s' - u^j) = \frac{\sqrt{f^j(u^j - s')}}{\sqrt{f^i(u^i - s)}} F_{ij}(s - u^i, s' - u^j) \tag{5.11}$$

the Zakharov differential relation (5.7) gives exactly $n(n-1)/2$ linear partial differential equations of the second order for $n(n-1)/2$ functions $\Phi_{ij}(s - u^i, s' - u^j)$ of two variables:

$$\frac{\partial}{\partial s'}\left(\frac{\sqrt{f^j(u^j - s')}}{\sqrt{f^i(u^i - s)}}\frac{\partial \Phi_{ij}(s - u^i, s' - u^j)}{\partial s}\right) - \frac{\partial}{\partial s}\left(\frac{\sqrt{f^i(u^i - s)}}{\sqrt{f^j(u^j - s')}}\frac{\partial \Phi_{ij}(s - u^i, s' - u^j)}{\partial s'}\right) \tag{5.12}$$

or, equivalently,

$$2\frac{\partial^2 \Phi_{ij}(s - u^i, s' - u^j)}{\partial u^i \partial u^j}\left(f^i(u^i - s) - f^j(u^j - s')\right) +$$

$$\frac{\partial \Phi_{ij}(s - u^i, s' - u^j)}{\partial u^j}\frac{df^i(u^i - s)}{du^i} - \frac{\partial \Phi_{ij}(s - u^i, s' - u^j)}{\partial u^i}\frac{df^j(u^j - s')}{du^j} = 0. \tag{5.13}$$

It is interesting that all these equations (5.13) for functions $\Phi_{ij}(s - u^i, s' - u^j)$ are the same and coincide with the corresponding single equation (4.3) for the two-component case.

Besides, for n functions $\Phi_{ii}(s - u^i, s' - u^i)$ we have also n linear partial differential equations of the second order from the Zakharov differential relation (5.7):

$$\frac{\partial}{\partial s'}\left(\frac{\sqrt{f^i(u^i - s')}}{\sqrt{f^i(u^i - s)}}\frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s}\right) + \frac{\partial}{\partial s}\left(\frac{\sqrt{f^i(u^i - s)}}{\sqrt{f^i(u^i - s')}}\frac{\partial \Phi_{ii}(s' - u^i, s - u^i)}{\partial s'}\right) \tag{5.14}$$

or, equivalently,

$$2\frac{\partial^2 \Phi_{ii}(s - u^i, s' - u^i)}{\partial s \partial s'}\left(f^i(u^i - s) - f^i(u^i - s')\right) -$$

$$\frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s}\frac{df^i(u^i - s')}{ds'} + \frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s'}\frac{df^i(u^i - s)}{ds} = 0. \tag{5.15}$$

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